Optimal Bid-Ask Spread in Limit-Order Books under Regime Switching Framework

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Abstract: This paper advocates a regime-switching model to capture the risk of structural changes in the economy, when determining the optimal bid-ask spread in limit order books. In our model, the market-maker faces an inventory risk due to the diffusive nature of the stocks’ mid-price and a transactions risk due to a Poisson arrival of market “buy” and “sell” orders. We propose that the intensity of the orders depend on the state of the economy, and in the event of a structural change, market makers will face different intensity of order flows, reflecting the new market conditions. In our model, the dealer is a wealth maximizing agent who dynamically manages his portfolio. We employ Hamilton-Jacobi-Bellman equation for the dynamic programming problem, and we solve the problem numerically using the Galerkin method.

JEL Classifications: C61, C02, D43, D58, G11
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1. Introduction

In the limit order book, a dealer is a specialist who is responsible for prioritizing top orders before the execution of other orders in the book, and before other orders at an equal or worse price held or submitted by other traders. For this specialized service, and the liquidity that is provided as a result, the dealer earns a spread between the bid and ask prices at which he is willing to buy and sell a specific quantity of an asset. There is a wealth of literature for different approaches to the determination of the bid-ask spread, none of which consider the regime switching risk. The numerous instances of financial distress, in local and global markets over the last fifty years, have proved that the market participants are exposed to the risk of structural changes in the economy. In particular, security dealers may suffer from the temporal imbalance in the equilibrium price of the securities, created by the changes in the state of the economy and reflected in the intensity of the buy or sell order flows. This paper is the first, which explicitly addresses this issue in the context of market microstructure.

In the Walrasian framework, an auctioneer aggregates traders' demands and supplies to find a market-clearing price. Through out the process of submitting demands and supplies, no actual trading occurs and each trader has the chance to revise his orders, until equilibrium is achieved. Modern security markets differ dramatically, as market participant trade in a continuum of time. Demsetz (1968) argues that there are additional costs for being involved in any market; which includes explicit costs such as the exchange fee, and implicit costs such as immediacy premium.
Immediacy premium is the cost that a trader is willing to pay for an early execution of a trade, which is particularly important as it extends the Walrasian model to the time dimension.\(^1\)

In this paper, we concentrate on the classic inventory models, introduced in T. S. Ho and Macris (1984). They found significant evidence of the inventory effect on the dealer's pricing strategy. The approach considers the economic agent as a rational utility maximizer; hence, the objective is not simply to trade but to manage the overall portfolio. We extend the model proposed by T. Ho and Stoll (1981), where they used the Bellman equation for the dynamic programming problem. In addition to the fact that we modulate some of the state variables with a Markov chain to capture the regime-switching framework, we propose a scientific numerical optimization method to find the optimal bid-ask spread. T. Ho and Stoll (1981) employed a verbal economic argument to postulate a solution for the Bellman equation. Most recently, Avellaneda and Stoikov (2008) found an approximate solution by performing a perturbation in inventory level. They have also demonstrated the practical importance of the inventory models. Our solution methodology is for a generic utility function, so it can be applied to practical situations with any utility function, with a very low implementation time.

Regime switching models are designed to capture the structural changes of the underlying parameters. The notion of regime switching economy fits well with the market microstructure approach to the equilibrium price. Nevertheless, there has been little attention to the impact of regime switching risk on market microstructure, amongst which are Chen, Diebold, and Schorfheide (2013), Valseth and Jørgensen (2011), Barberis, Shleifer, and Vishny (1998). The first two researchers focused on the empirical implications of the regime switch models, in particular on the data structure, and Barberis et al. (1998) investigates the problem from behavioral finance prospective. Moreover, Cartea and Jaimungal (2013) employed a hidden Markov model to examine how the intra-day dynamics of the stock market have changed, and how to use this information to develop trading strategies at high frequencies. They also solved the market making problem for exponential utility when both intensity and volatility are regime switching.

To the knowledge of the authors, there is no research in the literature that explicitly analyzes the theoretical modeling framework of finding the equilibrium price, when the economy is regime switching. The purpose of this paper is to develop a simple understanding about a) how may the equilibrium price change, if the economy changes regimes? We theorize that the structural changes in the economy may impact the magnitude of excess demand (supply) of securities in short run. b) How could a market maker adjust to the change, as a utility maximizer market participant?

We capture the effect of the changes in the state of the economy on the equilibrium price by modulating the intensity of the orders arrival with a Markov chain. For example, we expect a larger intensity for bid orders during the bear market, due to the larger excess supply of securities, compared to when the market is bullish. This would enable us not only to price the immediacy premium via setting the bid-ask spread, but also to investigate the changes in excess supply and excess demand under different states of the economy.

The idea of probability switching also appeared in the early development of nonlinear time series analysis, where Tong (1983) proposed the threshold time series models. Hamilton (1989) popularized regime-switching time series models in the economic and econometric literature. Due to the empirical success of regime-switching models, they have been applied to different important areas in banking and finance, including asset allocation, option valuation, risk management, term structure modeling, and others. Recently the spotlight seems to turn to option valuation under regime-switching models (see: Naik (1993), Guo (2001), Buffington and Elliott (2002), Elliott, Chan, and Siu (2005), Fard and Siu (2013), amongst others).

\(^1\) For a comprehensive survey of the literature, see O’hara (1995).
2. The Modeling Framework

Let us suppose that the economy can switch between a finite number of states, each of which are characterized by their respective parameters. In this economy, consider a financial market, where an agent can either invest in a risk-free money market account or choose from a range of risky assets. All the parameters of the risk-free asset, as well as the risky assets, vary as the economy switches regimes, a process governed by a Markov-chain.

We characterize the dealer as simply a market participant, amongst many. The dealer is assumed to be risk averse, and entitled to manage the risk and return of his portfolio of cash, inventory and base wealth position, according to his utility function. The dealer is compensated for the specialized service\(^2\) that he provides, and the risk that he bears by setting up the bid and ask price in a way that maximizes her terminal utility.

Consider a continuous-time financial model consisting of a stock and a money market account. In this market, we study the behavior of a single dealer making a market in a single stock. The assets are tradable continuously on a fixed time horizon \(T := [0, T]\), where \(T \in (0, \infty)\). We fix a complete probability space \((\mathcal{I}, \mathcal{F}, \mathbb{P})\), where \(\mathbb{P}\) is the real world probability measure. Let \(T\) denote the time index set \([0, T]\) of the economy. We describe the states of the economy by a continuous-time Markov chain \(\{X_t\}_{t \in T}\) on \((\mathcal{F}, \mathcal{F}, \mathbb{P})\), with a finite state space \(\mathcal{S} := \{s_1, s_2, ..., s_N\}\). Without loss of generality, we can identify the state space of the process \(\{X_t\}_{t \in T}\) to be a finite set of unit vectors \((e_1, e_2, ..., e_N)\), where \(e_i = (0, ..., 1, ..., 0) \in \mathbb{R}^N\).

Let \(A(t) = [a_{ij}(t)]_{i,j=1,2,..,N'} t \in T\), denote a family of generators, or rate matrices, of the chain \(\{X_t\}_{t \in T}\) under \(\mathbb{P}\). Here, \(a_{ij}(t)\) represents the instantaneous intensity of the transition of the chain \(\{X_t\}_{t \in T}\) from state \(j\) to state \(i\) at time \(t\). Note that for each \(t \in T\), \(a_{ij}(t) > 0\), for \(i \neq j\) and \(\sum_j a_{ij}(t) = 0\), so \(a_{ij}(t) \leq 0\). We assume that \(a_{ij}(t) > 0\), for each \(i, j = 1, 2, ..., N\) and \(i \neq j\) and each \(t \in T\). For any such matrix \(A(t)\), write \(a(t) := (a_{11}(t), ..., a_{ii}(t), ..., a_{NN}(t))'\).

The notation is adopted in Dufour and Elliott (1999). With the canonical representation of the state space of the chain, Elliott, Aggoun, and Moore (1994) provide the following semi-martingale decomposition for \(\{X_t\}_{t \in T}\):

\[
X_t = X_0 + \int_0^t A X_s ds + M_t
\]  

(1)

Here, \(M_t\) is a \(\mathbb{R}^N\)-valued martingale with respect to the filtration generated by \(\{X_t\}_{t \in T}\). Let \(\{r^f(t, X_t)\}_{t \in T}\) be the instantaneous market interest rate of a money market account, which depends on the state of the economy. That is, \(r^f := \langle r^f, X_t \rangle = \sum_{i=1}^N r^f_i \langle X_t, e_i \rangle \quad t \in T\)

(2)

where \(r^f := (r^f_1, r^f_2, ..., r^f_N)\) with \(r^f_i > 0\) for each \(i = 1, 2, ..., N\) and \(\langle , \rangle\) denotes the inner product in the space \(\mathbb{R}^N\).

Further, let \(\{Z_t\}_{t \in T}\) denote a standard Brownian motion on \((\mathcal{I}, \mathcal{F}, \mathbb{P})\) with respect to the \(\mathbb{P}\)-augmentation of its natural filtration \(\mathcal{F}^Z := \{\mathcal{F}^Z_t\}_{t \in T}\). Let \(\mu_t\) and \(\sigma_t\) denote the drift and volatility of the market value of the reference asset, respectively. Then:

\(^2\) When the market is absolutely competitive, the cost of specialized service is minimized (see O'hara (1995)).
\[ \mu_t := \langle \mathbf{\mu}, X_t \rangle = \sum_{i=1}^{N} \mu_i \langle X_t, e_i \rangle \]  
\[ \sigma_t := \langle \mathbf{\sigma}, X_t \rangle = \sum_{i=1}^{N} \sigma_i \langle X_t, e_i \rangle \]  
(3)

where \( \mathbf{\mu} = (\mu_1, \mu_2, \ldots, \mu_N) \) and \( \mathbf{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N) \); \( \mu_i \in \mathbb{R} \) and \( \sigma_i > 0 \) for each \( i = 1, 2, \ldots, N \). We assume here that \( \mu_t \) and \( \sigma_t \) depend on the current economic state \( X_t \) only. Let \( S := \{ S_t | 0 \leq t \leq T \} \) be the mid-price process of stocks, where \( S_0 = s > 0 \) and \( s \) is a given constant. The dealer believes that the price process of the stock is defined according to:

\[ S_t = S_0 + \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u \]  
(4)

The mid-price is used to value the dealer’s asset at the end of the investment period. The dealer is committed to respectively buy and sell \( Q \) shares of stock at these prices, should he be hit or lifted by a market order. We assume that the bid and ask order flows are governed by a class of Markovian models known as Markov Modulated Poisson Process, herein MMPP, defined as a bivariate Markov process of the form \( \{(N_t, X_t) \}_{t \geq 0} \).

Transactions are assumed to evolve as a stationary continuous time stochastic jump process. As in T. Ho and Stoll (1981) assume that a different process is allowed for purchases and sales by the dealer. In addition the process of “bid” and “ask” are regime switching, corresponding to the Markov chain \( X \). It is quite natural to suggest the Poisson jump process, \( N_t \), for model transactions, as is the case in T. Ho and Stoll (1981). We propose to consider different intensities for purchase and/or sale order arrivals under different states of the economy. This provides us a convenient framework to explore the temporal variations in excess demand and excess supply. Under this framework, the dealer quotes the bid price \( p^b_t \) and the ask price \( p^a_t \) based on his opinion of the true value of the stock at time \( t \), denoted by \( S_t \). Here, bid and ask are modulated by the Markov chain. Mathematically, write

\[ p^a_t := \langle \mathbf{p}^a, X_t \rangle = \sum_{i=1}^{N} p^a_i \langle X_t, e_i \rangle \]  
\[ p^b_t := \langle \mathbf{p}^b, X_t \rangle = \sum_{i=1}^{N} p^b_i \langle X_t, e_i \rangle \]  
(5)

where \( \mathbf{p}^a := (p^a_1, p^a_2, \ldots, p^a_N)' \in \mathbb{R}^N \) and \( \mathbf{p}^b := (p^b_1, p^b_2, \ldots, p^b_N)' \in \mathbb{R}^N \) is the transpose of a matrix, or a vector. \( \langle \ldots \rangle \) is the scalar product in \( \mathbb{R}^N \); \( p^a_i \) (resp. \( p^b_i \)) is the ask price (resp. the “bid” price) set by the dealer at time \( t \) when he observes the Markov chain \( X_t = e_i \).

Suppose that there is no cost involved in continuously updating the limit orders. The priority of execution when large market orders get executed is determined by the shape of the limit order book, and the following distances:

\[ a_t = S_t + p^a_t \quad b_t = S_t - p^b_t \]

This allows us to assume market buy orders will “lift” our agent’s sell limit orders at Poisson rate \( \lambda^a_t \), a decreasing function of \( a_t \). With the same token, orders to sell stock will “hit” the agent’s buy limit order at Poisson rate \( \lambda^b_t \), a decreasing function of \( b_t \). Pursuantly, we let \( dq^a(X) \) and \( dq^b(X) \) to be the dealer sales and purchases, respectively, which are defined as:

\[ \sim 36 \sim \]
The Wealth of the Dealer

The dealer is a utility maximizing economic agent, who is committed to meet the order follows. He begins with no initial cash or inventory and hence holds only the initial portfolio; therefore, his incentive to trade is the sufficient compensation, resulting from the spread. The wealth of the dealer at any time is the summation of three components; namely the money market account, his inventory of the stocks and his base wealth. Following, the dynamics are detailed:

A) A money market account, from which the dealer funds his purchases, and in which the trader deposit his surplus cash. The balance in the fund earns (pays if negative) risk free interest rate \( r^f_t \). Hence the dynamics of this asset could be expressed as:

\[
dF_t = r^f_t F_t \, dt - (S_t - b_t)dq^b(X_t) + (S_t + a_t)dq^a(X_t).
\]

B) The dealer’s inventory, which consists of shares of the one stock in which he makes a market. The change in the value of the inventory account, \( l_t \), is

\[
dl_t = r^I_t l_t \, dt + S_t dq^b(X_t) - S_t dq^a(X_t) + l_t dZ^I_t,
\]

where \( r^I_t \) is the cash return of the inventory (i.e. dividend payments), and \( Z^I_t \) is a standard Brownian motion associated with the dealer’s inventory. In this setup we highlight two interesting features. First, inventory is always valued at the known intrinsic value of the stock, \( S_0 \), and not at the prices at which it actually trades. Hence, bid and ask prices play no role in the valuation of the inventory. Second, the value of the inventory does change due to both changes in size and changes in its value resulting from the diffusion term \( l_t dZ^I_t \) and drift \( r^I_t l_t dt \).

C) The dealer’s portfolio also includes base wealth, \( Y_t \) whose change is described by

\[
dY_t = r^Y_t Y_t \, dt + Y_t dZ^Y_t
\]

where \( r^Y_t \) is the drift of the base wealth, and \( Z^Y_t \) is a standard Brownian motion. The dealer is interested to determine the spread at time \( t \), in a way that his utility of the terminal wealth is maximized. We formulate the problem using the dynamic programming approach, employing Hamilton-Jacobi-Bellman (HJB) equation. Write the dynamics of the wealth process as the following semi-martingale from

\[
dW_t = \left( r^I_t l_t + r^f_t F_t + r^Y_t Y_t + \lambda^b_t + \lambda^a_t \right) dt + b_t d\tilde{q}^b(X_t) + a_t d\tilde{q}^a(X_t)
\]

\[+ l_t dZ^I_t + Y_t dZ^Y_t, \quad W_0 = w \in \mathbb{R}^k.
\]

where \( \{\tilde{q}^a(X_t)\} \) and \( \{\tilde{q}^b(X_t)\} \) are compensated Poisson processes (hence, martingale) and defined as follows, as the consequence of Doob-Meyer decomposition of a Poisson process

\[
\tilde{q}^a(X_t) = q^a(X_t) - \int_0^t \lambda^a_u \, du
\]

\[
\tilde{q}^b(X_t) = q^b(X_t) - \int_0^t \lambda^b_u \, du
\]
We assume that we are given a family \( \pi_t = (a_t, b_t) \) of admissible controls, contained in the set of controls \( \pi_t = (a_t, b_t) \) such that (9) has a unique strong solution and such that

\[
E \left[ \int_0^T U(W_t) | \mathcal{H}_T \right] < \infty, \tag{10}
\]

where \( \mathcal{H}_T \) is the enlarged filtration, as follows

\[
\mathcal{H}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^X.
\]

For \( \pi \in \mathcal{A} \), we define the performance functional \( J^\pi(t, W, X) \) by

\[
J^\pi(t, W, X) = \left[ \int_0^T U(W_t) | \mathcal{H}_T \right] \tag{11}
\]

Therefore, the stochastic control problem is to find \( V(t, W_t, X) \) and \( \pi^* \in \mathcal{A} \) such that

\[
V(t, W_t, X) = \sup_{\pi \in \mathcal{A}} (J^\pi(t, W, X)) = J^{\pi*}(t, W, X) \tag{12}
\]

In the stochastic control literature, the function \( V \) is called the value function, and \( \pi^* \) is called an optimal control. Under mild conditions, Markov controls can give just as good performance as the general adapted controls. In view of this, we restrict ourselves to consider only Markov controls.

**Theorem 1** For the Markovian control \( \pi \in \mathcal{A} \), the generator of (12) is given as:

\[
L^\pi V(t, W_t, X) = \frac{\partial V}{\partial t} + (r^t_1 I_t + r^t_2 F_t + r^t_3 Y_t + \lambda^b_t + \lambda^a_t) \frac{\partial V}{W_t} \\
+ \frac{1}{2} (l^t_1 + y^t_2 + \rho \bar{y}_t) \frac{\partial^2 V}{W_t^2} \\
+ \lambda^b(X_t)(V(W_{t-} + b_t, X_t) - V(W_{t-}, X_t)) \\
+ \lambda^a(X_t)(V(W_{t-} + a_t, X_t) - V(W_{t-}, X_t)) \\
+ (V, AX_t).
\]

Proof is provided in the appendix.

**Theorem 2** Suppose \( V(t, W_t, X_t) \) is a bounded function in \( C^2(S) \cap C(\tilde{S}) \) and \( \pi \in \mathcal{A} \) is a Markov control, such that:

\[
(L^\pi V)(t, W, X) \leq 0; \quad \forall W \in S, \forall \pi \in K \tag{14}
\]

Then \( V(t, W, X) \geq J^\pi(t, W, X) \) for all \( W \in S \). Moreover, if for each \( W \in S \) we have found \( \hat{\pi} \) such that:

\[
(L^\pi V)(t, W, X) = 0 \tag{15}
\]

then \( \hat{\pi} \) is a Markov control such that

\[
V(0, W = W_0, X = X_0) = J^\pi(0, W = W_0, X = X_0)
\]

and if \( \hat{\pi} = (\hat{a}_t^a, \hat{b}_t^b) \in \mathcal{A} \) then \( \hat{\pi} \) must be an optimal control.

**Proof**. Assume that \( V(.) \) satisfies (15), and \( \pi \) is a Markov control. Hence, for all \( R < \infty \) we have:

\[
J^\pi(t, W, X) = E \left[ \int_0^T U(W_t) | \mathcal{H}_T \right] \\
\sim 38 \sim
\]
= \lim_{R \to \infty} E \left[ \int_0^{T_R} V(t, W, X) \mid \mathcal{F}_T \right] \\
= V(0, W = W_0, X = X_0) + E \left[ \int_0^{T_R} (\mathcal{L}V)(t, W, X) \right] \quad \text{(Dynkin’s Formula)} \\
= V(0, W = W_0, X = X_0) \quad \text{(As per equation (15)).} \\

3. Numerical Optimization and Solution

Here we assume that the dealers’ goal is to maximize the expected utility of his terminal wealth by setting the optimal bid-ask spread. The HJB equation of this problem can be written as:

\[
\begin{align*}
\sup_{\pi} (\mathcal{L}V(s, W_s, X)) &= 0 \\
V(T, W_T, X) &= E[U(\gamma, W_T) \mid \mathcal{F}_T] = U(\gamma, W_T),
\end{align*}
\]

where $\gamma$ to be the coefficient of the absolute risk aversion.

In this section we employ an elegant and computationally fast numerical scheme; namely, the Galerkin method, to find the optimal policy as well as the solution for (16). The Galerkin method is a generalized Finite Element Method (FEM, henceforth) and is an advantageous alternative to the Finite Difference Method (FDM, henceforth), which is the most commonly used numerical method in the literature. As a case in point, the Galerkin method uses the weak form formulation of the governing equations. The use of integral formulations is advantageous as it provides a more natural treatment of Neumann boundary conditions as well as that of discontinuous source terms due to their reduced requirements on the regularity or smoothness of the solution. Moreover, the solution found by the Galerkin method belongs to the entire domain, as opposed to the isolated nodes as in the case of FDM. Finally, complex geometries in multi-dimensional problems are handled with higher precision and higher computational speed, since the integral formulations do not rely on any special mesh structure.

The FEM is growing in popularity in the finance literature (e.g. Barth (2010), Holmes, Yang, and Zhang (2012), Forsyth, Vetzal, and Zvan (1999)). In particular, Holmes et al. (2012) employs the methodology in valuation of American option with regime switching.

In this section we first present the variational formulation for (16), which discretizes the space variable $W$ and transforms the PDE into a system of liner ODEs. During the process of the variational formulation, we separate the space variable from the time variable in the truncated domain of the value function. This allows us to approximate the control variables with a high level of precision, since the Ritz condition for the residuals is met for the entire domain. Consequently, we plug the optimal values for the bid and ask price back to the variational formulation of (16) and solve the value function.

The first step towards the numerical optimization and finding the solution $V$ is to truncate the space dimension, $W$, and impose the following generic boundary conditions.

\[
\begin{align*}
\alpha_1 V(W^{\text{min}}) + \beta_1 \frac{\partial V}{\partial W}(W^{\text{min}}) &= \theta_1, \\
\alpha_2 V(W^{\text{max}}) + \beta_2 \frac{\partial V}{\partial W}(W^{\text{max}}) &= \theta_2
\end{align*}
\]

The Galerkin method can easily incorporate boundary conditions involving derivatives, so the dealer has the freedom to choose $\alpha_i$ and $\beta_i$, as per their particular circumstances. Additionally, in
this specific optimization problem, the time-space domain of the parabolic problem is compact by itself.

3.1 Variational Formulation

One of the ingredients of the Galerkin method is the variational formulation. Combining the previous arguments, we re-write the problem as

\[
\begin{align*}
\frac{\partial V}{\partial t} + D V &= 0, \quad \forall (t, V) \in (0, T] \times \mathbb{R}, \\
\alpha_1 V(W^{\text{min}}) + \beta_1 \frac{\partial V(W^{\text{min}})}{\partial W} &= \theta_1, \quad \forall (t, V) \in \mathbb{R}, \\
\alpha_2 V(W^{\text{max}}) + \beta_2 \frac{\partial V(W^{\text{max}})}{\partial W} &= \theta_2, \quad \forall (t, V) \in \mathbb{R}, \\
V(T, W_T, X) &= 1 - e^{-\gamma W_T},
\end{align*}
\]

where

\[
(DV)(W, X) = (r_t^f I_t + r_t^f F_t + r_t^f Y_t + \lambda_t^p + \lambda_t^q) \frac{\partial V(W, X)}{\partial W} + \frac{1}{2} (l_t^2 + Y_t^2 + \rho l_t Y_t) \frac{\partial^2 V(W, X)}{\partial W^2} + \langle V, MX_t \rangle \\
+ \lambda_t^p (V(W_t + b_t, X_t) - V(t, W_t, X_t)) + \lambda_t^q (V(W_t + a_t, X_t) - V(t, W_t, X_t))
\]

Suppose \( \Psi \) is the subspace of the following Sobolev space of first order:

\[
H^1(\mathbb{R}) := \{ \psi \in L^1_{\text{loc}}(\mathbb{R}) | \psi, \psi_W \in L^2(\mathbb{R}) \}
\]

where

\[
|| \psi ||_{H^k(\mathbb{R})} = \left( \int_1^T |V(W, t, X)| + |\sum_{i=1}^{k} \frac{\partial^i \psi}{\partial W^i} |^2 dx \right)^{1/2}
\]

Then for the trial function, \( \psi \in \Psi \), the bilinear form \( D \) from \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) into \( \mathbb{R} \), associated with special operator \( D \) is given by:

\[
D(V, \psi, X) = \int_{W_{\text{min}}}^{W_{\text{max}}} \frac{1}{2} (l_t^2 + Y_t^2 + \rho l_t Y_t) \frac{\partial V(W, X)}{\partial W} \frac{\partial \psi}{\partial W} dW \\
- \int_{W_{\text{min}}}^{W_{\text{max}}} (r_t^f I_t + r_t^f F_t + r_t^f Y_t) \frac{\partial \psi}{\partial W} dW \\
+ \int_{W_{\text{min}}}^{W_{\text{max}}} \left( \lambda_t^p (V(W_t + b_t, X_t) - V(t, W_t, X_t)) \right) \psi dW \\
+ \lambda_t^q (V(W_t + a_t, X_t) - V(t, W_t, X_t)) \psi \psi dW \\
- \frac{1}{2} (l_t^2 + Y_t^2 + \rho l_t Y_t) \frac{\partial V(W, X)}{\partial W} \psi \bigg|_{W_{\text{min}}}^{W_{\text{max}}}
\]

Now, the variational formulation of (17) reads: Find \( V(W, t, X) \in \Psi \) such that for every fixed time

\[
\int_{W_{\text{min}}}^{W_{\text{max}}} \frac{\partial V}{\partial t} \psi dW + D(V, \psi) = 0, \quad \forall \psi \in \Psi \text{ and } 0 < t < T.
\]
3.2 Space Discretization

Next we use (19) to make a discretization of the space dimension. We restrict the bilinear form into functions in $H^1_0(\Omega_\tau)$, where $\Omega_\tau = (W_{\min}, W_{\max})$. By doing this, we are seeking a solution approximation $\hat{V}(t, W, X_t = e_j) \in \Psi$ such that for every fixed time.

$$\int_{W_{\min}}^{W_{\max}} \frac{\partial \hat{V}}{\partial t} \psi dW + D(\hat{V}, \psi) = 0, \quad \forall \psi \in \Psi \text{ and } 0 < t < T. \quad (20)$$

Let the interval $[W_{\min}, W_{\max}]$ be divided into $N$ interval, and let $S_h \subset H^1_0(\Omega_\tau)$, be the space of price-wise linear functions with basis $\{\phi^N_k\}$, such that we can expand

$$\hat{V}(t, W, X) = \sum_{i=1}^{N} \xi_i(t, X) \phi_k(W, X) \quad (21)$$

The approach of separating the time variable and the space variable is known as method of Kantorovich, after the Nobel prize winner L. V. Kantorovich. As a result of applying the method, our problem will be reduced to a system of simultaneous Ordinary Differential Equations. The approach is known as semi-discrimination of the value function, and in this section a stiff system of ODEs is archived via the discretization of (16). This is sufficient for us to find the optimal policy $\pi^*$. We present the characteristic function that solves the non-homogenous system of ODEs, analytically. However, in the numerical analysis we will discretize the time variable, using the fourth order Runge-Kutte method. This approach is called the vertical method of lines. A less popular approach is to discretize time first and spatial variables in a second step, which is known as Rothe’s method.

We reduce the problem into an economy that can switch between only two states. This assumption is only for convenience in presentation, and once can easily extend the result to higher number of states. Using straight forward, but tedious calculations of substituting $\hat{V}(t, W, X_t)$ into (19) and choosing the trial functions $\psi = \phi_k; k = 1, 2, ..., N$ we obtain the following system of ODEs:

$$\begin{pmatrix}
K(X = e_1) + \frac{d}{dt} B(X = e_1) & N(X = e_1)
N(X = e_1) & K(X = e_1) + \frac{d}{dt} B(X = e_1)
\end{pmatrix}
\begin{pmatrix}
\xi(X = e_1)
\xi(X = e_1)
\end{pmatrix} =
\begin{pmatrix}
q(X = e_1)
q(X = e_1)
\end{pmatrix} \quad (22)$$

Here $\xi(X, t)$ denotes a vector holding the nodal values $\xi_i(X, t)$ and:

$$K_{kk}(X_t = e_i)\begin{cases} K_{11}(X_t = e_i) = \frac{\alpha_1(l_t^2 + Y_t^2 + \rho l_t Y_t)}{2\beta_1}, & \text{for } K_{11} \\
K_{NN}(X_t = e_i) = \frac{\alpha_2(l_t^2 + Y_t^2 + \rho l_t Y_t)}{2\beta_2}, & \text{for } K_{NN} \\
K_{ik}(X_t = e_i), & \text{otherwise}
\end{cases} \quad (23)$$

where

$$K_{kk}(X_t = e_i) = \left(\frac{d\phi_k}{dW} - C, \frac{d\phi_k}{dW}\right) - \langle a_{ii} \phi_k, \phi_k \rangle + \langle \phi_k, \lambda^b_{e_i}(\phi(W_t + b) - \phi(W_t)) + \lambda^b_{e_i}(\phi(W_t + a) - \phi(W_t)) \rangle,$$

and

$$C = r_f I_t + r_f Y_t + r_f F_t + \lambda^b_t + \lambda^b_t - \frac{1}{2} (l_t^2 + Y_t^2 + \rho l_t Y_t).$$

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Recall that, denote the rate matrices of the two state chain under .

The dealer is now able to determine the optimal policy, , for each state by maximizing (22). Since the time variable is separated from the special variable, the optimization would be straightforward; however, from the computational point of view, the choice of the basis function, , is critically important. A linear basis function would simplify the optimization to a bang-bang control problem, whereas, the optimal values would be functions of had a higher order basis been chosen. In the latter case, the accuracy of the approximation would be considerably enhanced, but the process would be more computing time expensive. Once the optimal policy was found, it may be substituted back to (22) so that the solution for could be calculated. Then we apply Cramer’s rule for the system of linear ODEs, to show that the characteristic equation will be:

\[
\det \begin{pmatrix}
B(X = e_i) + DB(X = e_i) & N(X = e_j) \\
N(X = e_i) & K(X = e_j) + DB(X = e_j)
\end{pmatrix} = 0
\] (27).

4. Numerical Experiment

In this section we use an example to illustrate our numerical analysis to set the bid-ask spread, when a market operates in a regime switching economy. For optimization of the control variables, as well as the solution of the value function, we follow the procedure explained in the previous section. However, instead of the final step (i.e. solving the ODEs using the characteristic equations), we employ a well known numerical scheme; namely, the fourth order Runge-Kutta. This is due to the fact that when the number of special grids increases, finding an analytical solution for the system of ODEs turns to a tedious and inefficient task; albeit possible. From the practical point of view it is important to program the procedures efficiently, so that the model can respond to the market advancement in a timely fashion.

We suppose that , where is interpreted as the intensity parameter of the transition of the Markov chain. Therefore, the generator matrix takes the following form:

\[
A = \begin{pmatrix}
-\eta & \eta \\
\eta & -\eta
\end{pmatrix}
\] (28).

It is standard to use to represent the entries in the rate matrix, and is adopted from Boyle and Draviam (2007). We suppose that the market maker has an exponential utility function with the absolute risk aversion coefficient equal to 1. Further, consider the dealer has one year time horizon and faces asymmetric demand and supply for the trading security. In our modeling framework, the intensity of the order flows must be decreasing functions of their respective orders (i.e. bid or ask). For simplicity, we use the following linear functions, adopted from T. Ho and Stoll (1981).
Here, \( k_t := (k_t, X_t), k := (k_1, k_2, ..., k_N) \) is the state dependent expected number of sale (or purchase) transactions, and \( \zeta_t := (\zeta_t, X_t), \zeta := (\zeta_1, \zeta_2, ..., \zeta_N) \) is determined to give a risk neutral spread. These two parameters are set to be constants in T. Ho and Stoll (1981); however, due to the state dependency in our model they become time dependent.

Now suppose that in the normal economy, regime 1, we have a close to symmetric demand; however, in the boom economy, regime 2, the ask orders sharply increase and the bid orders dramatically decrease. For our example we consider a simple case that \( \zeta = 100,000 \) for all scenarios. In addition,

\[
\kappa_t^a(e_1) = 2300, \kappa_t^a(e_2) = 3000, \kappa_t^b(e_1) = 2500, \kappa_t^b(e_2) = 2200.
\]

Further, we consider some specimen values for the model parameters.

\[
\begin{align*}
\rho_t^l(e_1) &= 0.15p.a., \rho_t^l(e_2) = 0.25p.a., \\
\rho_t^l(e_1) &= 0.12p.a., \rho_t^l(e_2) = 0.22p.a., \\
\rho_t^l(e_1) &= 0.05p.a., \rho_t^l(e_2) = 0.10p.a.,
\end{align*}
\]

These values are chosen to maintain the consistency and comparability of our results with that presented in T. Ho and Stoll (1981).

Technically, since \( I_t \) and \( Y_t \) are state-dependent, the correlation coefficient between them must be state-dependent as well. Nevertheless, due to the fact that the regime switching risk is of the systematic risk nature, we do not expect a significant variation for the correlation coefficient. Hence, for simplicity we assume \( \rho(Y, I) = 0.7 \).

For this example, we impose the following boundary condition.

\[
\beta_1 = \beta_2 = 0.5, \alpha_1 = \alpha_2 = 0.5, \theta_1 = 1, \theta_2 = 800, \\
W_{\text{min}}(e_1) = W_{\text{min}}(e_2) = 1, W_{\text{max}}(e_1) = 1200, W_{\text{max}}(e_2) = 1200.
\]

For the implementation, we first choose a linear basis function for \( \phi \). The choice of linear basis turns our feedback control problem to a bang-bang control problem, as the controllers turn to be constants with reference to the value function. As a result, we achieve a straight forward, and computationally fast optimization process. Figure 1 presents the spread calculated for two different states. We observe that there is an adjustment in the spread when the economy changes regime to the boom economy. In the boom economy, the market maker experiences a dramatic excess demand for the instrument.

From an economic point of view, when the intensity of the ask orders sharply increases, a rational utility maximizer dealer would opportunistically increase the ask price to take advantage of the strong demand. This theory is consistent with equations (6) and (7); in particular, when the intensity for the ask orders is dramatically higher than the bid orders, \( dI_t \) is expected to increase, in a short period of time. It is also noteworthy that figure 1 shows the spread is increased by 4\% in the regime 2. Finally, when the dealer is faced the regime-switching risk, they may set the bid-ask spread as the average of each state's spread with respect to the Markov chain's transition probabilities. The impact of this is illustrated more clearly in Figure 4.
Figure 1. The Bid-Ask spread under two regimes. In this example a linear basis function is used.

Figure 2. Bid-Ask spreads in regime 1 for different value functions, and 3 different choices of the basis function.

Figure 3. Bid-Ask spreads in regime 2 for different value functions, and 3 different choices of the basis function.
One advantage of the Galerkin method is that the accuracy of the numerical solution could be easily improved by choosing a higher order basis. From the point of view of our optimization problem, when we choose a higher order basis function, the optimal controls, $\pi^*$ is no longer constant with respect to the value function. Figure 2 presents the optimal spread against a range of different values for the value function, when three different basis functions are chosen; namely, linear, quadratic and cubic. It is important to report that the choice of higher order basis has a considerable impact on the computational expenses\(^3\); both from the time and the memory point of view. In our case, the optimal values were calculated 427% slower when the cubic basis was used, compared to the linear basis.

**Figure 4.** Expected utility of the dealer's wealth after one quarter. Both graphs are based on linear basis function.

We further analyze the value function, by comparing the result to the case without regime-switching risk. We assume that the parameter values of no-regime-switching version match with those in the corresponding regime-switching process when the economy is in state one. Figure 4 shows the expected utility of the agent one quarter after the starting point. The utility function continues to evolve as we move backwards more time steps. It will be bounded by the utility of a purely risk free strategy, a function given by multiplying the terminal utility by a $\exp (-\gamma t)$. The figure compares the expected utility under regime-switching scenario and no-regime-switching scenario. It can be viewed that the regime-switching scenario maximizes the utility for all different wealth levels.

**Figure 5.** Dealer's wealth over the investment horizon of one year. Both graphs are based on linear basis function.

Additionally, Figure 5 presents the dealer's accumulation of wealth over time under the two scenarios; namely, with and without regime-switching. Here, we consider the analysis for the expected utility equal to one, however, the results are easily extendable.

\(^3\)Recently, this literature has become very popular in High Frequency Data trading, where the computational expense is one of the core reservations of algorithms development.
5. Conclusion

This paper, for the first time in the literature, considers the risk of regime-switching economy in determination of bid-ask spread for a market maker. We use the classic inventory model, where the market maker is assumed to be a risk-neutral wealth maximizing investor. He is assumed to hold an inventory of the risky asset, whose price evolves according to the regime switching version of the diffusion process. He is also assumed to maintain a level of cash in the money-market account, whose value compounds as per the risk free rate determined in different states of the economy. Under this framework, the dealer receives bid and/or ask order flows, according to two Markov-switching Poisson processes. In particular, the intensities of the order flows are modulated by a finite-states Markov chain, to distinguish between the frequency of the orders received in different states of the economy. This, we believe, is an appropriate model for capturing the impact of excess demand and supply of different securities, raised from the structural changes in the economy.

We employ a dynamic programming approach to model the dealer's portfolio, however, due to the complexity of the generator function the close-form solution for the optimal bid-ask spread is not achievable. Therefore, we utilize an elegant subclass of Finite Element Method, namely the Galerkin method, to find the optimal control variables as well as solving the value function. To highlight the practical implications, we conduct a numerical example. We observe that the bid-ask spread chosen considering the regime switching risk increases the expected utility of the dealer, and maximizes the value of his portfolio of the risky asset.

References


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Appendix

A Proof of Theorem 1:

The wealth process can be written as \( W_t = W_t^c = W_t^d \), the former being the continuous part and the latter being the discontinuous part of the process. Apply Ito’s differential rule to (12).

\[
V(t, W_t, X_t) = V_0(0, W_0, X_0) + \int_0^t \frac{\partial V}{\partial u} du + \int_0^t \frac{\partial V}{\partial W} dW_t^c + \frac{1}{2} \int_0^t \frac{\partial^2 V}{\partial W^2} d\langle W_t, W_t \rangle
\]

\[
+ V(t, W_t^- + \Delta W_t, X_t) - V(t, W_t^-, X_t) + \int_0^t \langle V, dX_t \rangle
\]

Since the process \( W \) consists of two poisson processes; namely, \( q_i^t = Q\lambda_i^t, i \in \{a, b\} \), we shall write

\[
V(t, W_t, X_t) = V_0(0, W_0, X_0) + \int_0^t \frac{\partial V}{\partial u} du + \int_0^t \frac{\partial V}{\partial W} dW_t^c + \frac{1}{2} \int_0^t \frac{\partial^2 V}{\partial W^2} d\langle W_t^c, W_t^c \rangle
\]

\[
+ \lambda^b(X_t)[V(t, W_t^- + b_t Q, X_t) - V(t, W_t^-, X_t)]
\]

\[
+ \lambda^a(X_t)[V(t, W_t^- + a_t Q, X_t) - V(t, W_t^-, X_t)] + \int_0^t \langle V, dX_t \rangle
\]

Recall equation (9), then

\[
dW_t^c = (r_t^Y l_t + r_t^Y F_t + r_t^Y Y_t + \lambda^b + \lambda^a) dt + l_t dZ_t^l + Y_t dZ_t^Y,
\]

\[
d\langle W_t^c, W_t^c \rangle = (dW_t^c)^2 = (l_t^2 + Y_t^2 + \rho l_t Y_t) dt,
\]
with $\rho$ being the correlation coefficient between the two, so that:

$$W_t^\xi = \rho Z_t^\eta + \sqrt{1 - \rho^2} Z_t^\eta$$

Besides, equation (1) provides the semi-martingale decomposition of the chain $X$, hence

$$\int_0^t \langle V, dX_u \rangle = \int_0^t \langle V, AX_u \rangle du + \int_0^t \langle V, dM_u \rangle,$$  \hspace{1cm} (A32)

where the second term is a martingale. Here, $V = (V_1, V_2, ..., V_N)$. Then plug (2) into (1).

$$V(t, W, X) = V_0(0, W_0, X_0) + \int_0^t \frac{\partial V}{\partial u} du + \int_0^t \frac{\partial V}{\partial W} dW_t^\xi + \frac{1}{2} \int_0^t \frac{\partial^2 V}{\partial W^2} d\langle W_t^\xi, W_t^\xi \rangle$$

$$+ \lambda^b(X_t)[V(t, W_{t-} + b, X_t) - V(t, W_{t-}, X_t)]$$

$$+ \lambda^a(X_t)[V(t, W_{t-} + a, X_t) - V(t, W_{t-}, X_t)]$$

$$+ \int_0^t \langle V, AX_u \rangle du + \int_0^t \langle V, dM_u \rangle$$

Our aim is to derive the generator $\mathcal{L}V(t, W_t, X_t)$, which is defined as:

$$\mathcal{L}V(t, W_t, X_t) = \lim_{t \to 0^+} \frac{1}{t} \{ E[V(t, W_t, X_t)] - V(0, W_0, X_0) | W_0 = w, X_0 = x \}$$

$$= \frac{d}{dt} E[V(t, W_t, X_t) | W_0 = w, X_0 = x]$$

We start with calculating $E[V(t, W_t, X_t)]$ by taking the expectation of (3):

$$E[V(t, W_t, X_t)] = V_0(0, W_0, X_0) + E \left[ \int_0^t \frac{\partial V}{\partial u} du \right] + E \left[ \int_0^t \frac{\partial V}{\partial W} dW_t^\xi \right]$$

$$+ \frac{1}{2} E \left[ \int_0^t \frac{\partial^2 V}{\partial W^2} d\langle W_t^\xi, W_t^\xi \rangle \right] + \lambda^b(X_t)E[V(t, W_{t-} + b, X_t) - V(t, W_{t-}, X_t)]$$

$$+ \lambda^a(X_t)E[V(t, W_{t-} + a, X_t) - V(t, W_{t-}, X_t)]$$

$$+ E \left[ \int_0^t \langle V, AX_u \rangle du \right].$$

Then we take the differential and (13) follows.