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Empirical Analysis of the Mexican Pension Funds Returns: Their Univariate and Multivariate Probability Distributions

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Abstract: This paper studies the probability distributions of the pension returns funds of the workers in the private sector, called *SIEFORES* (SB) in Mexico. Till September 2013, there were five pension funds (*SIEFORES*) and the daily returns of each one are reported as the average return obtained by the private companies (managers, called *AFORES*) in the corresponding fund. We estimate the parameters of the univariate density function through maximum likelihood and the goodness of fit using the Kolmogorov test. In the multivariate part, parameters are estimated with an EM algorithm and goodness of fit of the distribution is developed using a simple test for continuous multivariate distributions of any dimension. The results show that the Normal Inverse Gaussian distribution is suitable for the adjustment of such multivariate returns data.

JEL Classifications: C16, G11,G23

Keywords: SIEFORES, Normal Inverse Gaussian distribution, Pension fund

1. Introduction

As mentioned in (Devolder, Janssen, and Manca, 2012): "A pension scheme can be defined as a systematic and organized mechanism, prescribed by law or by a convention, in order to provide after retirement regular incomes to a well defined category of people". In the Mexican case, the congress is the responsible for the design of the mechanism and the *CONSAR* (Mexican Pension Fund Regulatory Agency) takes care for the application of the designed rules.

Institutional investors like pension funds have important challenges trying to design investment portfolios that should be financially sustainable in the long term. As mentioned in (Villaseñor-Zertuche, 2000), the principal variables defining the amount of wealth at the end of the labor cycle of an agent are: size of contributions, time contributing and the return.

In the case of Mexico, the design of the pension policy in the congress is related with the periodic size of contributions and the time contributing. On the other hand, the return depends on the financial management of each individual agent fund and there are some private financial companies (pension fund managers) whose objective is the investment of such agents' resources. The investment management companies (private) are called AFOREs and the agents´ funds are called SIEFORES (SB).

Before the reform of 1997 of the pension funds system in Mexico, IMSS (Mexican Social Security Institute, one of the two big institutions of health), has controlled since 1944 the funds of the Mexican workers in the private sector in a pay-as-you-go system; in this kind of system pensions there are problems because of the demographic changes (the rates of fertility and mortality have fallen), health costs and not enough actuarial reserves to sustain the level of benefits (Grandolini and Cerda, 1998). The original IMSS system was designed as a partially funded defined benefit system, but as mentioned in (Espinosa-Vega M.A. and Tapen S.,2000) in practice was operating as a pay as you go system. The new system is a defined contribution pension scheme: resources are given by employee, employer and government and then invested by the managers (called AFORES in Mexico) of the pension funds.

Until September 2013, depending on the age of the worker, there were five possible retirement funds (SIEFORES) for the investment of the resources: SB1, SB2, SB3, SB4 and SB5. For example, if the worker is 28 years old, he must be in the SB4. We show the table 1 below with the classification by age. Once the worker knows his/her corresponding fund, he can select the manager in which can invest his/her resources.

Table 1. Workers are divided in five different funds depending on the age

Name of Fund (SIEFORES)	**Age (years) of the worker**
SB1	At least 56
SB2	46-55
SB3	37-45
SB4	27-36
SB5	At most 26

Source: *CONSAR,* http://www.consar.gob.mx

Strategies of investment are critical for the future wealth of the population. Moreover, it has been studied that there is a statistical connection of the pension funds resources with macro-economic variables and therefore we are not just talking about the welfare of the individual workers but the macroeconomic environment (Villagómez and Antón, 2013) and (Corbo, Schmidt-Hebbel). There are studies with models to calculate the fund performance; in the case of (Sinha, Martínez and Barrios-Muñoz 1999) a recursive model to estimate the future value of the individual worker fund is proposed. However we consider that it is necessary a discussion about the information in the historical returns of the SIEFORES; for example, suppose that the returns distribution is heavy tailed and therefore potentially pension funds can affect seriously the macroeconomic environment, if a loss occurs. In the defined contribution system in the Mexican case, there have been important changes in the kind of instruments in which investment is permitted (Rodríguez L. Jacobo, 1999).

In this paper, for the data of returns we estimated an empirical probability distribution which belongs to the Generalized Hyperbolic family, in this case the Normal Inverse Gaussian distribution. The consequences for the risk management and the possible regulatory policies of the government are a fundamental issue. The return is the result of the strategies of investment of the pension funds managers and therefore a statistical study of them is necessary.

In this paper we study the univariate and multivariate distribution of the five SIEFORES in Mexico. In section 2 we review the data with the estimation procedure and the basic notions of the Generalized Hyperbolic Family. In section 3 we present the conclusions.

2. Data and Estimation Procedure

We have calculated the returns of each *SIEFORE* (as mentioned above there exist five of these type of SIEFORE depending on the age of the contributor) and such returns were calculated using the formula,

$$r_i = \frac{P_i}{P_{i-1}} - 1$$

where each P_i is reported in the electronic direction of *CONSAR*. In this way we have five time series and the period of time is from July 1997 to September 2013. It is important to mention that the SIEFORES are composed of several managers funds, and the price reported by CONSAR is an average of the prices of each manager belonging to each SIEFORE.

The hyperbolic generalized family has been used for the adjustment of the stock returns, see (Barndorff-Nielsen, 1995). This family of distributions has interesting particular cases depending on the range of variation of the parameters, (Blaesild, 1990); for example two important cases are the hyperbolic distribution as in (Eberlein and Keller, 1995) for λ=1 and the Normal Inverse Gaussian (NIG) distribution for λ=-1/2.

There is a lot of evidence about the non-normality of the returns in the stock exchange of different countries; in the Mexican case we can see this fact in (Trejo, Núñez and Lorenzo, 2006). Because of the great adjustment flexibility, the generalized hyperbolic distribution has been used in different financial empirical studies, see (McNeil, Frey and Embrechts, 2005). The univariate generalized hyperbolic distribution (HG) has different parameterizations, see for example (Paolella, 2007). The usual specification is defined as,

$$f(x;\mu,\alpha,\delta,\beta,\lambda) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})} K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu))$$

where K_v is the modified Bessel function of the third kind and is given by

$$K_v(x) = \frac{1}{2}\int_0^{\infty} w^{v-1}\exp\left(\left[-\frac{1}{2}x(w+w^{-1})\right]\right) dw$$

for $x > 0$, (Abramowitz and Stegun, 1972).

The domain of the parameters is: $\lambda, \mu, \beta \in \mathbb{R}$ and $\alpha, \delta \geq 0$. In an important way, the case $\lambda = -1/2$ has been used to adjust successfully some time series of stock returns, (Rydberg, 1999).

The multivariate NIG distribution for a random vector X of size $n \times 1$ is

$$f(x;\lambda,\gamma,\beta,\mu,\delta,\Delta) = \frac{\left(\sqrt{\dfrac{\delta^2 + (x-\mu)\Delta^{-1}(x-\mu)'}{\gamma^2 + \beta'\Delta\beta}}\right)^{\lambda-\frac{n}{2}}}{(2\pi)^{n/2}\left(\frac{\gamma}{\delta}\right)^{-\lambda}} \frac{K_{\lambda-\frac{n}{2}}\left(\sqrt{\delta^2 + (x-\mu)\Delta^{-1}(x-\mu)'(\gamma^2+\beta'\Delta\beta)}\right)}{K_\lambda(\delta\gamma)\exp\left(-(x-\mu)\beta'\right)}$$

The estimation is made by the algorithm in (Protassov, 2004), where the EM algorithm is employed on the two basic steps:

Step 1: Calculate the expected value of the likelihood function by the starting point Θ_0 and the random sample $x_1, x_2, \dots, x_m$. Thus each kth iteration obtain the objective function

$$g\left(\Theta;\Theta^{[k]}\right) = E\left[\ln\left(L'(\Theta;\ x_1,\dots,x_n,\mathrm{w}_1,\mathrm{w}_2,\dots,\mathrm{w}_\mathrm{n})\right)|x_1,\dots,x_m;\Theta^{[k]}\right]$$

Step 2: We maximize the objective $g\left(\Theta;\Theta^{[k]}\right)$ function to get the next approximation $\Theta^{[k+1]}$.

The initial values to start the iterations are $\mu = \alpha = 0$, $\delta = \gamma = 1/2$, $\Delta = I$ the identity matrix and β a vector with 1's.

In this document we check the goodness of fit of the multivariate NIG distribution using a simple test for continuous multivariate distributions of any dimension, (McAssey, 2013).

This empirical algorithm basically consists of the following steps:

a) We assume that the null hypothesis is true, it means that the multivariate distribution from SIEFORES is NIG and we estimate via the EM algorithm the corresponding parameters.

b) We generate a sample $u_1, u_2, \dots, u_N$ with size 10000 by simulation via the parameters found in a).

c) We calculate the Mahalanobis distance

$$\hat{d}_i^{\,2} = (\hat{u}_i - \hat{\mu})'\hat{\Delta}^{-1}(\hat{u}_i - \hat{\mu})$$

for the series in part b) using the mean $\hat{\mu}$ and the covariance matrix $\hat{\Delta}$.

d) We select a partition $\{p_0, p_1, \dots, p_T\}$ in the interval $[0,1]$ with $p_0 = 0$, $p_T = 1$, and we estimate the p-quantile from

$$\hat{G}_N(t) = \sum_{i=1}^{N} I\left(\hat{d}_i \le t\right)/N$$

where $t \in \left(0{,}2max\{\hat{d}_i\}\right)$ and $q_0 = 0$, $q_j = min\left\{t \in \mathbb{R} | p_j \le \hat{G}_N(t)\right\}$ in $j = 1,2,\dots,T-1$ and $q_T = \infty$.

e) We found $E_j = n(p_j - p_{j-1})$ and count O_j observations from the succession $\hat{d}_i$ in the interval $(q_{j-1}, q_j]$ for $j = 1,2,\ldots,T-1$ and in the interval $(q_{T-1}, \infty]$.

f) Finally we calculate the test statistic

$$A_T = \sum_{j=1}^{T} |1 - \frac{O_j}{E_j}|$$

Empirically we find the p value using a simulation of sample size M on adjusted multivariate distribution function, (McAssey, 2013), and we reject the null hypothesis if the p-value is less than some given level of significance,

$$p - value = \frac{1}{M}\sum_{i=1}^{M} I\left(A_{T,i} > A_T\right).$$

In table 2 we exhibit the correlations matrix of the different returns, which shows an important linear dependence among the returns. Also, in this paper we estimated the parameters of the NIG univariate function through maximum likelihood procedure for the returns of the five *SIEFORES*. The goodness of fit is investigated using the Kolmogorov test considering the NIG distribution and the normal distribution, see table 3 and figure 1. It can be observed that the null hypothesis (data follow a NIG distribution) cannot be rejected and on the other hand the hypothesis of normal distribution is rejected. In conclusion there is evidence to confirm that the returns do not follow a normal distribution and do not reject that the distribution is Normal Inverse Gaussian.

Table 2. Correlations for the return of the five SIEFORES, 2008-2013

	SIEFORE SB 1	SIEFORE SB 2	SIEFORE SB 3	SIEFORE SB 4	SIEFORE SB 5
SIEFORE SB 1	1.000	0.941	0.923	0.903	0.889
SIEFORE SB 2	0.941	1.000	0.998	0.992	0.983
SIEFORE SB 3	0.923	0.998	1.000	0.998	0.991
SIEFORE SB 4	0.903	0.992	0.998	1.000	0.994
SIEFORE SB 5	0.889	0.983	0.991	0.994	1.000

Table 3. Parameters estimation of the NIG distribution and goodness of fit for the returns of *SIEFORES* 2008-2013

SIEFORE	**Parameters estimation: NIG (λ = -1/2)**				**Goodness of fit (p-value)**	
	μ	δ	α	β	NIG	Normal
Siefore SB 1	0.0004	0.0018	207.1523	-6.9890	0.7401	0.0000
Siefore SB 2	0.0005	0.0022	116.4658	-9.5852	0.3640	0.0000
Siefore SB 3	0.0006	0.0027	100.6100	-8.9947	0.6361	0.0000
Siefore SB 4	0.0007	0.0033	91.6107	-9.3058	0.5763	0.0000
Siefore SB 5	0.0006	0.0034	85.0925	-5.9373	0.7172	0.0000

In table 3 above, we compare the normal distribution and our adjustment of the Normal Inverse Gaussian distribution (empirical adjustment). All the 5 cases show a strong deviation from normality.

Estimation of the multivariate Normal Inverse Gaussian distribution for SIEFOREs SB1, SB2, SB3, SB4 and SB5 is presented in Table 4. The corresponding p-value for the test of (McAssey, 2013) is 0.2067, which indicates not rejection of the null hypothesis. In other words, there is no evidence to reject the multivariate NIG distribution for different SIEFORES.

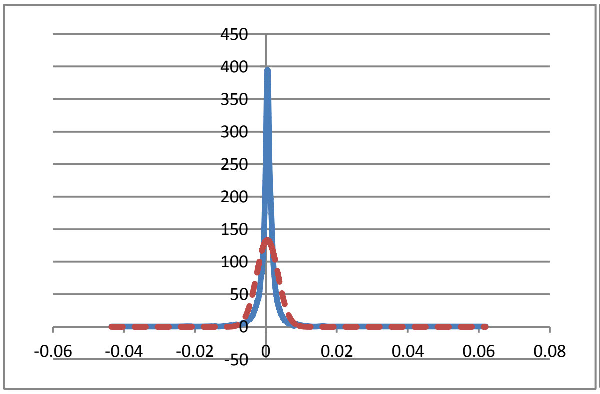
Table 4. Estimation via the EM algorithm with λ = -1/2

Iterations	56
p-value	0.2067
Log-likelihood	37061.37
λ	-0.5000
δ	0.0027
γ	120.1863

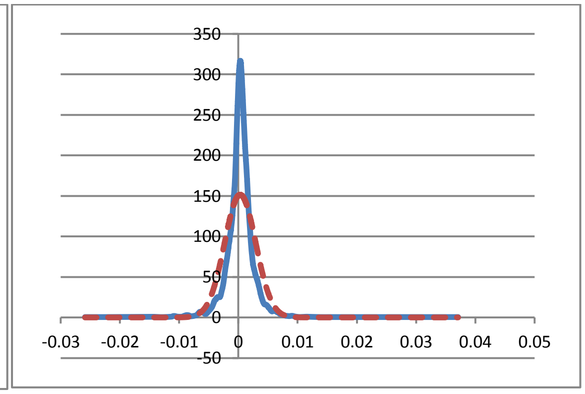
Matrix **Δ**

SIEFORE	Siefore SB 1	Siefore SB2	Siefore SB 3	Siefore SB 4	Siefore SB 5
Siefore SB 1	0.00000845	0.00001037	0.00001191	0.00001348	0.00001417
Siefore SB 2	0.00001037	0.00001573	0.00001868	0.00002188	0.00002316
Siefore SB 3	0.00001191	0.00001868	0.00002237	0.00002636	0.00002802
Siefore SB 4	0.00001348	0.00002188	0.00002636	0.00003131	0.00003335
Siefore SB 5	0.00001417	0.00002316	0.00002802	0.00003335	0.00003597

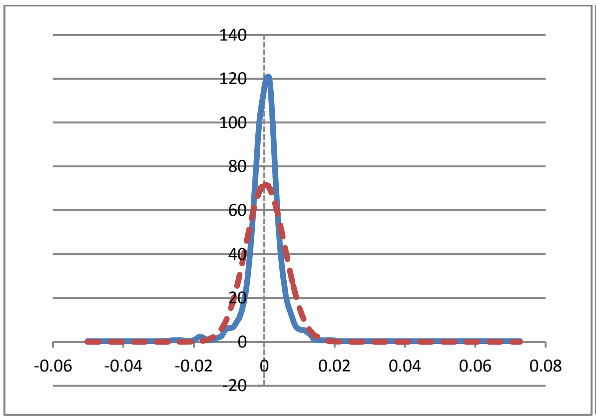
SIEFORE	Siefore SB 1	Siefore SB 2	Siefore SB 3	Siefore SB 4	Siefore SB 5
β	-0.0000443	-0.0001175	-0.0001564	-0.0001996	-0.0002848
μ	0.0003490	0.0004241	0.0004825	0.0005509	0.0005991



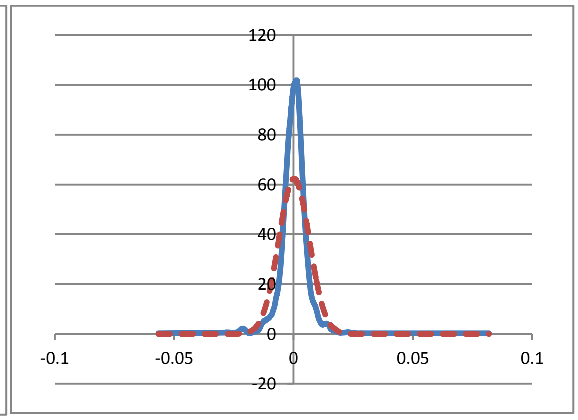
SIEFORE SB1



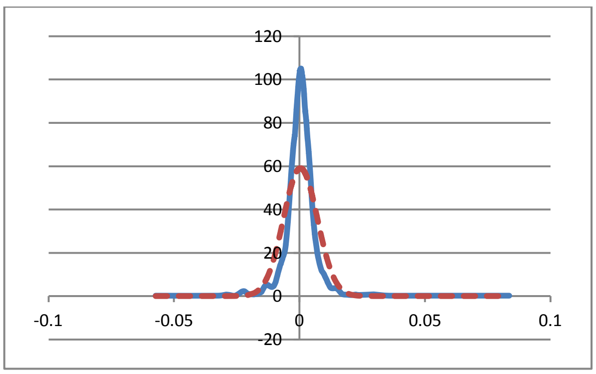
SIEFORE SB2



SIEFORE SB3



SIEFORE SB4



SIEFORE SB5

Figure 1. Probability distributions of the *SIEFORES*
(Dotted line corresponds to the Normal distribution and the solid line represents the empirical distribution, in each graph.)

3. Conclusions

This paper concludes that

(1) The Multivariate Normal Inverse Gaussian probability distribution has a reasonable goodness of fit to all SIEFORES SB1, SB2, SB3, SB4 and SB5.

(2) A similar finding is found through the classical Kolmogorov test for the univariate probability distributions of each individual SIEFORE returns with the Normal Inverse Gaussian distribution.

These facts indicate the relevance of heavy tails in daily returns of SIEFORES in their empirical distributions, and therefore this kind of information is important for the practitioners of the risk management of the pension funds.

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